On the nonlinear rational difference equation

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Abstract— We investigate some qualitative behavior of the solutions of the difference equation $x_{n+1} = ax_{n-\ell} + (bx_{n-k} / cx_{n-s} + dx_{n-k})$ where the initial conditions $x_{-r}, x_{-r+1}, \ldots, x_0$ are arbitrary positive real numbers such that $r = \max\{\ell, k, s\}$ where $i, r \in \{0, 1, \ldots\}$ and a, b, c, d are positive constants.

keywords — difference equation, Stability, Periodicity, boundedness, global Stability.

1 Introduction

In this paper we deal with some properties of the solutions of the difference equation

$$x_{n+1} = ax_{n-\ell} + (bx_{n-k} / cx_{n-s} + dx_{n-k}), \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where the the initial conditions $x_{-r}, x_{-r+1}, \ldots, x_0$ are arbitrary positive real numbers such that $r = \max\{\ell, k, s\}$ where $i, r \in \{0, 1, \ldots\}$ and a, b, c, d are positive constants. There is a class of nonlinear difference equations, known as the rational difference equations, each of which consists of the ratio of two polynomials in the sequence terms in the same form. there has been a lot of work concenring the global asympototic of solutions of rational difference equations [1], [2], [3], [5], [6], [8], [10] and [11].

Many reseaches have investigated the behavior of the solution of difference equation for example:

Stevic [11] has studied the stability, global attractor, and the periodic character of solutions of the equation

$$x_{n+1} = \alpha x_n + (x_{n-1} / x_n).$$

Our aim in this paper is to extend and generalize the work in [11]. That is, we will investigate the global behavior of (1.1) including the asymptotical stability of equilibrium points, the existence of bounded solution, and the existence of period two solution of the recursive sequence of Eq. (1.1)

Now we recall some well-known results, which will be useful in the investigation of (1.1) and which are given in [7].

Let I be an interval of real numbers and let

$$F : I^{k+1} \to I,$$

where F is a continuous function. Consider the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

with the initial condition $x_{-k}, x_{-k+1}, \ldots, x_0 \in I.$

Definition 1 (Equilibrium Point)

A point $x \in I$ is called an equilibrium point of Eq. (1.2) if

$$x = f(x, x, \ldots, x).$$

That is, $x_n = \bar{x}$ for $n \geq 0$, is a solution of Eq. (1.2), or equivalently, $\bar{x}$ is a fixsed point of f.

Definition 2 (Stability)

Let $x \in (0, \infty)$ be an equilibrium point of Eq. (1.2). Then

(i) An equilibrium point x of Eq. (1.2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-r}, x_{-r+1}, \ldots, x_0 \in (0, \infty)$ with $|x_{-r} - x| + |x_{-r+1} - x| + \ldots + |x_0 - x| < \delta$ then $|x_n - x| < \varepsilon$ for all $n \geq -r$.

(ii) An equilibrium point $\bar{x}$ of Eq. (2) is called locally asymptotically stable if $\bar{x}$ is locally stable and there exists $\gamma > 0$ such that, if $x_{-r}, x_{-r+1}, \ldots, x_0 \in (0, \infty)$ with $|x_{-r} - \bar{x}| + |x_{-r+1} - \bar{x}| + \ldots + |x_0 - \bar{x}| < \gamma$, then

$$\lim_{n-\infty} x_n = \bar{x}.$$

(iii) An equilibrium point $\bar{x}$ of Eq. (1.2) is called a global attractor if for every $x_{-r}, x_{-r+1}, \ldots, x_0 \in (0, \infty)$ we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iv) An equilibrium point $\bar{x}$ of Eq. (1.2) is called globally asymptotically stable if x is locally stable and a global attractor.

(v) An equilibrium point $\bar{x}$ of Eq. (1.2) is called unstable if x is not locally stable.

Definition 3 (Permanence)

Eq. (1.2) is called permanent if there exists numbers m and M with $0 < m < M < \infty$ such that for any initial conditions $x_{-r}, x_{-r+1}, \ldots, x_0 \in (0, \infty)$ there exists a positive integer N which depends on the initial conditions such that

$$m \leq x_n \leq M \quad \text{for all } n \geq N.$$

Definition 4 (Periodicity)

A sequence $\{x_n\}_{n=-r}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -r$. A sequence $\{x_n\}_{n=-r}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

The linearized equation of Eq. (1.2) about the equilibrium point x is defined by the equation

$$z_{n+1} = \sum_{i=0}^{k} p_i z_{n-i} = 0, \tag{1.3}$$

where

$$p_i = \partial F(\bar{x}, \bar{x}, \ldots, \bar{x},) / \partial x_{n-i}, \quad i = 0, 1, \ldots, k.$$

The characteristic equation associated with Eq. (1.3) is

$$\lambda^{k+1} - p_0 \lambda^k - p_1 \lambda^{k-1} - \ldots - p_{k-1}\lambda - p_k = 0. \tag{1.4}$$

Theorem 1.1 [7] Let $[a, b]$ be an interval of real numbers and assume that

$$f : [a,b]^3 \to [a,b]$$

is a continuous function satisfying the following properties:

$f(u,v,w)$ is non-increasing in $w \in [a,b]$ for each u and $v \in [a,b]$, and is non-decreasing in u and $v \in [a,b]$ for each $w \in [a,b]$.

If $(m,M) \in [a,b] \times [a,b]$ is a solution of the system

$$m = f(m,m,M) \quad \text{and} \quad M = f(M,M,m),$$

implies

$$m = M.$$

Theorem 1.2 [7]. Assume that F is a C^1 $-$ function and let $\bar{x}$ be an equilibrium point of Eq. (1.2). Then the following statements are true:

If all roots of Eq. (1.4) lie in the open unit disk $|\lambda| < 1$, then he equilibrium point $\bar{x}$ is locally asymptotically stable.

If at least one root of Eq. (1.4) has absolute value greater than one, then the equilibrium point $\bar{x}$ is unstable.

If all roots of Eq. (1.4) have absolute value greater than one, then the equilibrium point $\bar{x}$ is a source.

Theorem 1.3 [9] Assume that $p_i \in R, i = 1,2,\ldots,k.$ Then

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptoticcally stable of Eq. (1.5)

$$y_{n+k} + p_1 y_{n+k-1} + \ldots + p_k y_n = 0, \; n = 0,1,\ldots. \qquad (1.5)$$

2 Local Stability of the Equilibrium Point

In this section we investigate the local stability character of the solutions of Eq. (1.1). Eq. (1.1) has a unique nonzero equilibrium point

$$\bar{x} = a\bar{x} + (b\bar{x}/(c+d)\bar{x}),$$

if $(1-a) > 0,$ then the only positive equilibrium point of Eq. (1.1) is given by

$$\bar{x} = (b/(c+d)(1-a)).$$

Let $f : (0,\infty)^3 \to (0,\infty)$ be a function defined by

$$f(u,v,w) = au + (bv/bw + dv). \qquad (2.1)$$

Therefore it follows that

$$\partial f(u,v,w)/\partial u = a,$$

$$\partial f(u,v,w)/\partial v = (bcw/(cw+bv)^2),$$

and

$$\partial f(u,v,w)/\partial w = (-bcv/(cw+bv)^2).$$

Then we see that

$$\partial f(\bar{x},\bar{x},\bar{x})/\partial u = a = -c_0,$$

$$\partial f(\bar{x},\bar{x},\bar{x})/\partial v = (c(1-a)/(c+d)) = -c_1,$$

and

$$\partial f(\bar{x},\bar{x},\bar{x})/\partial w = (-c(1-a)/(c+d)) = -c_2.$$

Then the linearized equation of (1.1) about $\bar{x}$ is

$$z_{n+1} = \sum_{i=0}^{k} p_i z_{n-i}. \qquad (2.2)$$

Theorem 2.1 Assume that

$$c < d.$$

Than the equilibrium point of Eq. (1.1) is locally stable.

Proof. It is follows by Theorem(1.3) that, Eq. (2.2) is locally stable if

$$|p_2| + |p_1| + |p_0| < 1.$$

This implies that

$$a(c+d) + c(1-a) + c(1-a) < c+d,$$

then

$$2c(1-a) < (c+d)(1-a).$$

Thus

$$c < d.$$

Hence, the proof is completed.

Example 2.1 Consider the difference equation

$$x_{n+1} = 0.1x_{n-\ell} + (3x_{n-k}/x_{n-s} + 2x_{n-k}),$$

where $a = 0.1, b = 3, c = 1, d = 2.$ Figure(2.1), shows that the equilibrium point of Eq. (1.1) has locally stable, with initial data $x_{-2} = 1.2, x_{-1} = 1.1, x_0 = 0.5.$

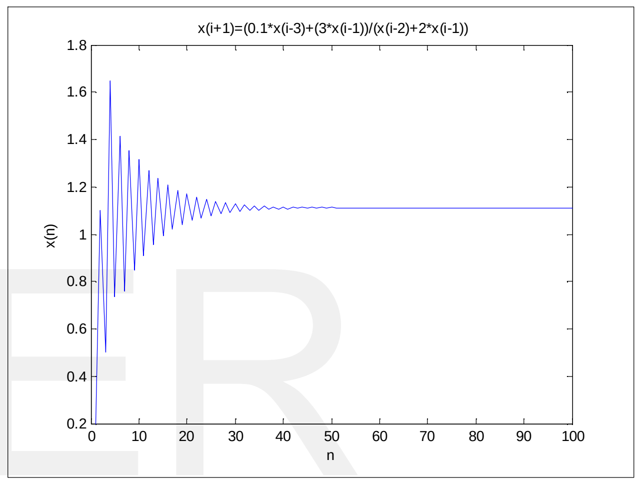

Figure 2.1

3 Periodic Solutions

Theorem 3.1 Eq. (1.1) has positive prime priod two solution only If

$$(i) \;\; \ell, s - even, k - odd \;\; and \;\; (c-d)(1+a) > 4ac. \qquad (3.1)$$

Proof. Assume that there exists a prime period-two solution

$$\ldots, p, q, p, q, \ldots$$

of (1.1). Let $x_n = q, x_{n+1} = p.$ Since $\ell, s - even, k - odd$, we have $x_{n-\ell}, x_{n-s} = q, x_{n-k} = p$. Thus, from Eq. (1.1), we get

$$p = aq + (bp/cq + dp),$$

and

$$q = ap + (bq/cp + dq).$$

Than

$$cpq + dp^2 = acq^2 + adpq + bp, \qquad (3.2)$$

and

$$cpq + dq^2 = acp^2 + adpq + bq. \qquad (3.3)$$

Subtracting (3.2) from (3.3) gives

$$d(p^2 - q^2) = ac(q^2 - p^2) + b(p-q).$$

Since $p \neq q$, we have

$$p+q=(v/d+ac). \qquad (3.4)$$

Also, since p and q are positive again, adding (3.2) and (3.3) yields

$$2cpq+d(p^2+q^2)=ac(p^2+q^2)+2adpq+b(p+q). \qquad (3.5)$$

It follows by (3.4), (3,5) and the relation

$$p^2+q^2=(p+q)^2-2pq, \quad \forall p,q\in \mathrm{R},$$

that

$$pq=(b^2ac/(d+ac)^2(c-d)(1+a)). \qquad (3.6)$$

It is clear now, from Eq. (3.4) and Eq. (3.6) that p and q are the two distinct roots of the quadratic equation

$$t^2-(b/d+ac)t+(b^2ac/(d+ac)^2(c-d)(1+a))=0,$$

and so

$$(b/d+ac)^2-(4b^2ac/(d+ac)^2(c-d)(1+a))>0,$$

which is equivalent to

$$(c-d)(1+a)>4ac.$$

Hence, the proof is completed.

Example 3.1 Consider the difference equation

$$x_{n+1}=0.125x_{n-\ell}+(x_{n-k}/0.5x_{n-s}+0.4x_{n-k}), \text{ where}$$
$$\ell,s-even,k-odd,a=0.125,c=0.5,b=1,d=0.4.$$

Figure(3.1), shows that Eq. (1.1) which is periodic with period two. Where the initial data satisfies condition(3.1) of Theorem(3.1) $x_{n-2}=0.2, x_{-1}=1.3, x_0=0.5.$ (seeTable 3.1)

n	$x(n)$	n	$x(n)$	n	$x(n)$	n	$x(n)$	n	$x(n)$
1	0.2000	17	0.3146	33	0.3169	49	0.3169	65	0.3169
2	1.3000	18	1.3761	34	1.3851	50	1.3852	66	1.3852
3	0.5000	19	0.3152	35	0.3169	51	0.3169	67	0.3169
4	1.6479	20	1.3854	36	1.3851	52	1.3852	68	1.3852
5	0.3951	21	0.3159	37	0.3169	53	0.3169	69	0.3169
6	1.2319	22	1.3885	38	1.3851	54	1.3852	70	1.3852
7	0.3562	23	0.3165	39	0.3169	55	0.3169	71	0.3169
8	1.1844	24	1.3885	40	1.3852	56	1.3852	72	1.3852
9	0.3330	25	0.3168	41	0.3169	57	0.3169	73	0.3169
10	1.2195	26	1.3874	42	1.3852	58	1.3852	74	1.3852
11	0.3224	27	0.3169	43	0.3169	59	0.3169	75	0.3169
12	1.2769	28	1.3864	44	1.3852	60	1.3852	76	1.3852
13	0.3171	29	0.3170	45	0.3169	61	0.3169	77	0.3169
14	1.3244	30	1.3857	46	1.3852	62	1.3852	78	1.3852
15	0.3149	31	0.3170	47	0.3169	63	0.3169	79	0.3169
16	1.3570	32	1.3853	48	1.3852	64	1.3852	80	1.3852

Table 3.1

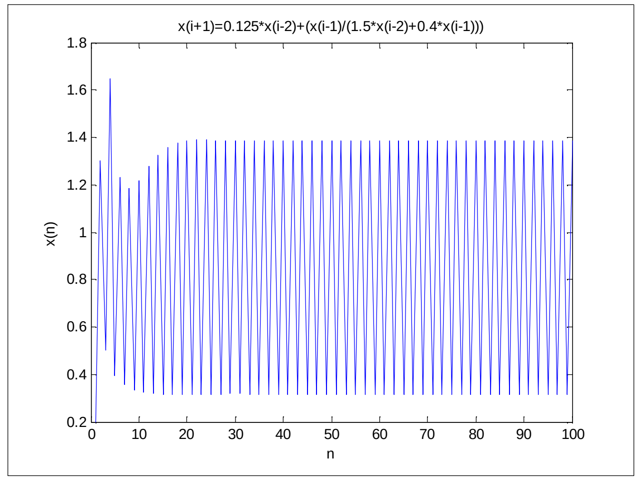

Figure 3.1

4 Bounded Solution

Our aim in this section we investigate the boundedness of the positive solutions of Eq. (1.1).

Theorem 4.1 For Eq. (1.1) every solution is bounded if $1>a.$

Proof. Let $\{x_n\}_{n=-r}^{\infty}$ be a solution of Eq. (1.1). It follows from Eq. (1.1) that

$$x_{n+1}=\le ax_{n-\ell}+(bx_{n-k}/dx_{n-k})=ax_{n-\ell}+(b/d) \quad \text{forall } n\ge 1.$$

By using a comparison, we can write the right hand side as follows

$$y_{n+1}=ay_{n-\ell}+(b/d),$$

then

$$y_n=a^n y_{-\ell}+(b/d),$$

and this equation is locally stable because $1>a,$ and converges to the equilibrium point

$$y=(b/d(1-a)).$$

Therefore

$$\lim_{n\to\infty}\sup x_n\le b/d(1-a).$$

Thus, for Eq. (1.1) every solutoin is bounded and the proof is completed.

Theorem 4.2 For Eq. (1.1) every solution is unbounded if $1<a.$

Proof. Let $\{x_n\}_{i=-r}^{\infty}$ be a solution of Eq. (1.1). It follows from Eq. (1.1) then

$$x_{n+1}=ax_{n-\ell}+(bx_{n-k}/cx_{n-s}+dx_{n-k})>ax_{n-\ell} \quad \text{for all } n\ge 1.$$

We can written as follows

$$y_{n+1}=ay_{n-\ell},$$

then

$$y_n=a^n y_{-\ell}, \qquad (4.1)$$

and the Eq. (4.1) is unstable because $1<a,$ and

$$\lim_{n\to\infty} y_n=\infty.$$

Thus, the proof is completed.

Example 4.1 Consider the difference equation

$$x_{n+1} = 0.8x_{n-\ell} + (2x_{n-k} / x_{n-s} + 2x_{n-k}),$$

where $k = 1, \ell = 2, s = 2, a = 0.8, b = 2, c = 1, d = 2.$ Figure(4.1), shows that the Eq. (1.1) has bounded, with initial data $x_{-2} = 0.3, x_{-1} = 0.2, x_0 = 0.1.$

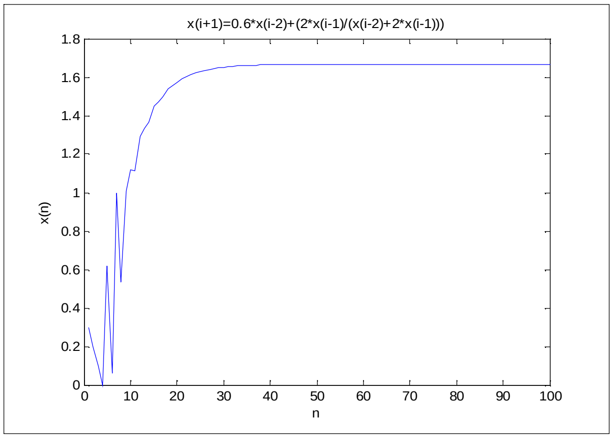

Figure 4.1

Example 4.2 Consider the difference equation

$$x_{n+1} = x_{n-\ell} + (2x_{n-k} / x_{n-s} + 2x_{n-k}),$$

where $k = 1, \ell = 2, s = 2, a = 1, b = 2, c = 1, d = 2.$ Figure(4.2), shows that the Eq. (1.1) has unbounded, with initial data $x_{-2} = 0.3, x_{-1} = 0.2, x_0 = 0.1.$

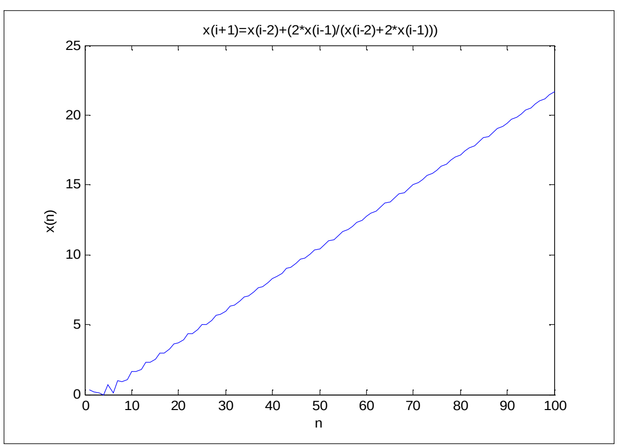

Figure 4.2

5 Global Attractor of the Equilibrium point of Eq. (1.1)

This section is devoted to investigate the global attractivity character of solutions of Eq. (1.1)

Theorem 5.1 if $a < 1$ and $c < d$ then the equilibrium point $\bar{x}$ of Eq. (1.1) is global attractor.

Proof. Let $f : (0,\infty)^3 \to (0,\infty)$ be a function defined by Eq. (2.1). Therefore

$$\partial f(u,v,w)/\partial u = a,$$

$$\partial f(u,v,w)/\partial v = (bcw/(cw+dv)^2),$$

and

$$\partial f(u,v,w)/\partial w = (-bcv/(cw+dv)^2).$$

Than we can easily see that the function $f(u,v,w)$ increasing in u, v and decreasing in w. suppose that (m, M) is a solution of the system

$$m = f(m,m,M) \quad \text{and} \quad M = f(M,M,m).$$

Then from Eq. (1.1).we see that

$$m = am + (bm / cM + dm), \quad M = aM + (bM / cm + dM)$$

$$m(1-a) = (bm / cM + dm), \quad M(1-a) = (bM / cm + dM),$$

then

$$(1 - a/b) = (1/cM + dm), \quad (1 - a/b) = (1/cm + dM).$$

Thus

$$M = m.$$

It follows by Theorem(1.1) that $\bar{x}$ is a global attractor of Eq. (1.1) and then the proof is complete.

Remark 5.1 Note that the special cases of Eq. (1.1) have been studied in [11] when

$$k = 1, \ell = 0, s = 0, a = \alpha, b = 1, c = 1, d = 0.$$

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